2022 Summer Internship in Mathematics: Lattices, Sphere Packing and Modular Forms

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1 Introduction

This research project investigated the relationship between sphere packing problems, lattices, and modular forms, as discussed in Noam Elkies' paper 'Lattices, Linear Codes, and Invariants, Part 1'. [Elk00] This report begins by giving a brief description of the sphere packing problem, which motivates what follows. The abstract mathematical structure of a lattice is then introduced, and its connection to the sphere packing problem by way of the simplifying 'lattice packing' assumption is explained. The densest lattice packing in $\mathbb{R}^2$ is described and its optimality with respect to density is proved. In the final section, the theta function for a lattice is defined and it is briefly outlined why theta functions for some lattices are examples of modular forms - a type of analytic function widely studied in branches of mathematics such as number theory which initially appear unrelated to the sphere packing problem.

2 The Sphere Packing Problem

The sphere packing problem generalizes certain problems which arise quite naturally from the study of the physical geometry of familiar two- and three-dimensional spaces and can be stated easily and intuitively.

- Given an unlimited supply of identical coins to be placed on a surface such as a table top, what percentage of the table top can be covered by the coins if no two ever overlap?
- How can balls be arranged in a three-dimensional box so as to fit the maximum possible number of balls in the box? (This is a version of a problem posed by Kepler, who sought to identify the most space-efficient way of stacking cannonballs.) [Elk00]

Abstracting these problems, both can be regarded as seeking an arrangement of identical balls in $\mathbb{R}^n$ for $n = 2, 3$ where the interiors of no two balls overlap and the greatest possible proportion of some convex region of $\mathbb{R}^n$ is covered by

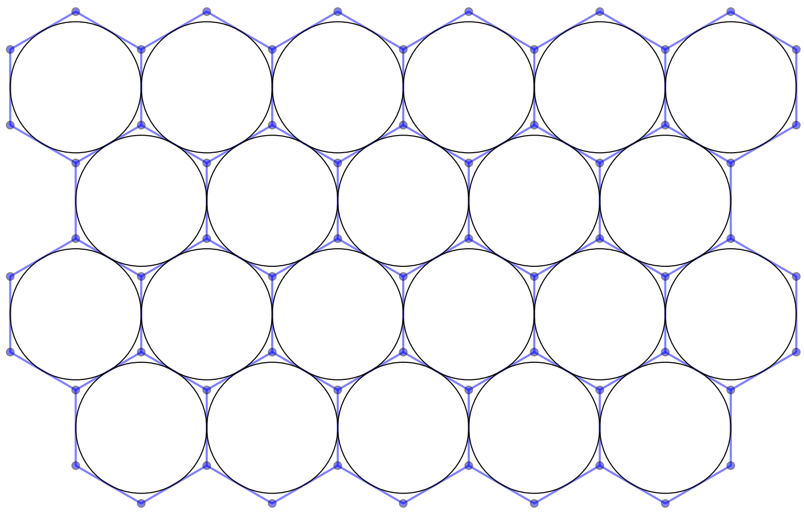

Figure 1: The hexagonal lattice packing is the densest possible in $\mathbb{R}^2$

the balls. Although physical intuition is lacking for $n = 4, 5, ...$, a similar mathematical problem can be posed for any arbitrarily high number of dimensions. A **sphere packing** is an arrangement of infinitely many n-dimensional of radius r with non-overlapping interiors in $\mathbb{R}^n$. (The term 'sphere packing' was coined before modern mathematical terminology which regards a ball as a volume in n-dimensional and a sphere as its surface became current.) For convex subsets of $\mathbb{R}^2$ it can be shown that the maximum proportion of the subset that can be covered by a sphere packing approaches a limit, L as the volume of the subset approaches infinity. This L is defined as the **maximum sphere packing density** for $\mathbb{R}^n$. The packing which achieves this density will be referred to as the **densest packing** or **optimal packing with respect to density**. The sphere packing problem in n dimensions asks what the value of the maximum sphere packing density is in $\mathbb{R}^n$

The sphere packing problem problem is a difficult one and despite intense study over many decades it has only been solved for four values of n, specifically $n = 2$ [Fej42], $n = 3$ [Hal+17], $n = 8$ [Via17], and $n = 24$ [Coh+17]. Even for the comparatively 'easy' case of a sphere packing in $\mathbb{R}^2$, the densest packing was only proved optimal in a rigorous manner in the 1940's, although a description of this packing had been known since antiquity and it had long been conjectured to have maximum density. (The densest packing for $n = 2$ is the 'hexagonal packing', pictured in Figure 1 and which will be discussed in more detail in Section 4). For the remaining cases where a best packing is known, a proof optimality has been provided only in the twenty-first century.

In the four non-trivial cases where it is known, the densest packing is a special type called a 'lattice packing', in which the centres of the balls form a structure called a lattice in $\mathbb{R}^n$. It is a somewhat easier problem to find

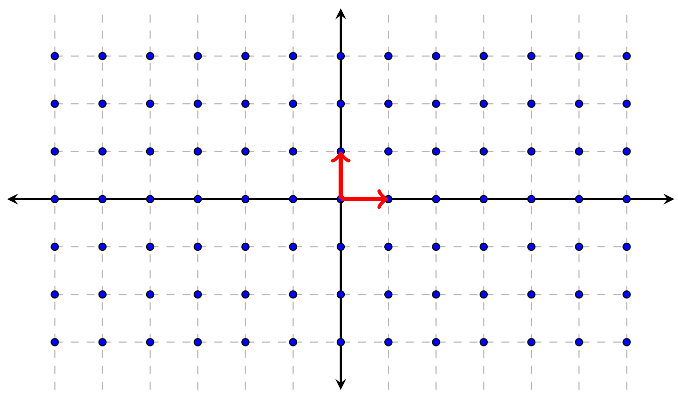

Figure 2: The integer coordinate lattice in $\mathbb{R}^2$, with standard basis vectors.

the maximum packing density among lattice packings than that for all possible packings. The fact that these densest lattice packings have turned out to be optimal among all packings for all dimensions in which an optimal packing is known gives a rationale for focusing on this easier class of problems as a first attempt at tackling general sphere packing problem. Further study lattice packings natural requires that lattices themselves be descibed in some degree of detail, and this is the purpose of the next section.

3 Lattices

3.1 Definition and Basic Example

Lattices may be defined for any vector space, but for the purposes of this report we consider only those in Euclidean space, $\mathbb{R}^n$, for arbitrary n

Definition 3.1 (Lattice)**.** Given a collection of linearly independent vectors, $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_n} \in \mathbb{R}^n$ (i.e. a basis of $\mathbb{R}^n$), the set of all integer linear combinations of $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_n} \in \mathbb{R}^n$ is a lattice in $\mathbb{R}^n$, denoted Γ. Thus:

$$\Gamma := \{y \in \mathbb{R}^n : y = \sum_{i=1}^{n} k_i x_i \mid k_1, ..., k_n \in \mathbb{Z}\}$$

Example 3.1. The set of all vectors with integer coordinates in $\mathbb{R}^n$ is clearly a lattice, as it comprises all integer linear combinations of the standard basis vectors. Figure 2 illustrates this lattice for the n = 2 case.

3.2 Generator Matrices

A second important definition is that of a generator matrix of a lattice.

Definition 3.2 (Generator Matrix of a Lattice)**.** Given a lattice $\Gamma \subset \mathbb{R}^n$, if the lattice consists of integer linear combinations of basis $\overrightarrow{x_1} = \langle x_{1,1}, ..., x_{n,1}\rangle, ..., \overrightarrow{x_n} = \langle x_{1,n}, ..., x_{n,n}\rangle$, then the matrix A_Γ such that:

$$A_\Gamma := [x_{i,j}]_{1 \leq i,j \leq n} \in \mathbb{R}^{nxn}$$

is a generator matrix for Γ. (A generator matrix is thus a matrix whose columns are basis vectors which are combined in integer linear combination to generate lattice members.)

Remark 3.1. If A_Γ is the generator matrix for lattice Γ, it is clear from the above definitions that we can express Γ as follows:

$$\Gamma = \{A_\Gamma \overrightarrow{z} \mid \overrightarrow{z} \in \mathbb{Z}^n\}$$

Example 3.2. For the lattice of integer coordinate vectors in $\mathbb{R}^n$, the nxn identity matrix is a generator matrix.

Any lattice in $\mathbb{R}^n$ can be represented by an invertible nxn generator matrix, and any invertible matrix defines a lattice. However distinct matrices do not necessarily represent distinct lattices - for example, rearranging the columns of a matrix gives a different matrix which generates the same lattice. Other cases of distinct matrices generating the same lattice exist as well. Fortunately, there is a simple criterion which determines whether two matrices generate the same lattice, the statement of which requires another definition.

Definition 3.3 (Integer General Linear Group)**.** The integer general linear group of degree n, $GL_n(\mathbb{Z})$, is a subset nxn matrices with integer coordinates defined as follows:

$$GL_n(\mathbb{Z}) := \{M \in \mathbb{Z}^{nxn} \mid det(M) = \pm 1\}$$

$GL_n(\mathbb{R})$ is defined analogously. $SL_n(\mathbb{Z})$ and $SL_n(\mathbb{R})$, the special linear group of $\mathbb{Z}$ and $\mathbb{R}$, consist of matrices whose determinet is $+1$.

Theorem 3.1. *Invertible nxn matrices A, B generate the same lattice if and only if there exists $M \in GL_n(\mathbb{Z})$ such that $B = AM$*

Proof. Let A, B be invertible matrices such that $\overrightarrow{v_{A,1}}, ..., \overrightarrow{v_{A,n}}$ are the columns of A and $\overrightarrow{v_{B,1}}, ..., \overrightarrow{v_{B,n}}$ are the columns of B. Also, let $a_{i,j}$ be the (i, j)$^{\text{th}}$ entry in A and $b_{i,j}$ be the (i, j)$^{\text{th}}$ entry in B. The lattice generated by A consists of all integer linear combinations of the linear independent vectors $\overrightarrow{v_{A,1}}, ..., \overrightarrow{v_{A,n}}$ and the lattice generated by B consists of all integer linear combinations of $\overrightarrow{v_{B,1}}, ..., \overrightarrow{v_{B,n}}$.

Now assume matrices A, B generate the same lattice, Γ. Then for $1 \leq j \leq n$: $\mathbf{v_{B,j}} = B\mathbf{e_j}$ where $\mathbf{e_j} \in \mathbb{Z}^n$ is the vector with 1 as the j$^{\text{th}}$ coordinate and 0 for all other coordinates. Hence, by Remark 3.1, $\overrightarrow{v_{B,j}} \in \Gamma$.

A is also a generator matrix for Γ, so for all $1 \leq j \leq n$, Definition 3.2 implies:

$$\exists m_{1,j}, ..., m_{n,j} \in \mathbb{Z} : \overrightarrow{v_{B,j}} = \sum_{k=1}^{n} m_{k,j} \overrightarrow{v_{A,k}}$$

Considering $\overrightarrow{v_{B,j}}$ entry-wise, this implies:

$$\forall 1 \leq i, j \leq n : b_{i,j} = \sum_{k=1}^{n} m_{k,j} a_{i,k} = \sum_{k=1}^{n} a_{i,k} m_{k,j}$$

And this means $B = AM$ where M is the nxn matrix $M = [m_{k,j}]_{1 \leq k,j \leq n}$. Further, $M \in \mathbb{Z}^{nxn}$ as it was assumed $m_{k,j} \in \mathbb{Z}$ for all k, j.

A similar argument proves there must an integer valued matrix K such that $BK = A$. Then $(BK)M = B$ and so there must exist $M^{-1} = K$.

It is a basic result that $Det(M^{-1}) = \frac{1}{Det(M)}$. However, both M and $M^{-1} = K$ are integer valued matrices, and therefore their determinants must both be integers (as the Laplace Expansion Formula for Determinant expresses the determinant as a sum of products of matrix entries). $Det(M)$ and $\frac{1}{Det(M)}$ can only both be integers if $Det(M) = \pm 1$. Hence $M \in GL_n(\mathbb{Z})$ and this proves the 'only if' part of the theorem statement.

For the 'if' part of the theorem, assume:

$$\exists M \in GL_n(\mathbb{Z}) : B = AM$$

Letting $m_{i,j}$ be the (i, j)$^{\text{th}}$ entry of M, for all j such that $1 \leq j \leq n$:

$$\overrightarrow{v_{B,j}} = \sum_{i=1}^{n} m_{i,j} \overrightarrow{v_{A,i}}$$

As every $m_{i,j}$ is an integer, it follows that all integer linear combinations of the columns of B, $\overrightarrow{v_{B,j}}$ can be expressed as integer linear combination of columns of A. Therefore, by Definition 3.2, the lattice generated by B is contained in the lattice generated by A. However, as $GL_n(\mathbb{Z})$ is a group, if $M \in GL_n(\mathbb{Z})$, then $M^{-1} \in GL_n(\mathbb{Z})$, and $A = BM^{-1}$. Thus it can similarly be shown that the lattice generated by A is contained in the lattice generated by B. Hence the lattices generated by A and B are in fact the same. completing the 'if' part of the proof. $\square$

A simple but important property of lattices is that of minimum distance.

Definition 3.4 (Minimum Distance of a Lattice)**.** For lattice $\Gamma \in \mathbb{R}^n$, the minimum distance of Γ, $m(\Gamma)$ is defined:

$$m(\Gamma) := \{Min(|\gamma_1 - \gamma_2|)\}_{\gamma_1 \neq \gamma_2 \in \Gamma}$$

where $|x|$ is the Euclidean length of vector $x \in \mathbb{R}^n$

The lattice packing associated with a lattice Γ is the sphere packing that results when a ball of radius $\frac{1}{2}m(\Gamma)$ is placed at each point of the lattice. It is clear that such an arrangement satisfies the constraint placed on a sphere packing that no two balls can overlap.

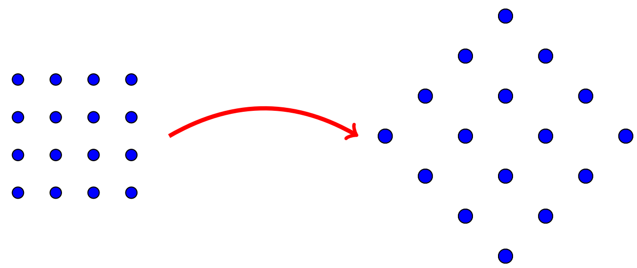

Figure 3: Homothetic lattices related by rotation and constant multiplication.

3.3 Homothecy

Generator matrices offer a concise way of describing lattices. It is therefore of interest to investigate what features of a lattice packing can be deduced from the generator matrix of the associated lattice. It is of especially fundamental importance to determine when two generator matrices give rise to lattices with the same lattice packing density. Loosely speaking, homothecy is a relationship between lattices defined such that homothetic lattices give rise to packings of the same density.

Obviously, if two generator matrices generate the same lattice then the density of the lattice packing associated with each of them is the same. By Theorem 3.1, then, a lattice with generator matrix B has the same packing density as lattice with generator matrix A if there exists $M \in GL_n(\mathbb{Z})$ such that $B = AM$. However, this does not exhaust the ways in which lattice packings can have the same density.

If the lattice generated by B is obtained by rotating all vectors in the lattice generated by A through some fixed angle about the origin, it is clear the associated lattice packings must have the same density, even though the lattices A and B are different. Rotation in $\mathbb{R}^n$ is represented by left multiplication by members of the special orthogonal group of $\mathbb{R}$ of dimension n, $SO_n(\mathbb{R})$. Thus generator matrices A and B generate lattices with the same packing density if there exists $U \in SO_n(\mathbb{R})$ such that $B = UA$

Furthermore, if all vectors in a lattice are multiplied by a positive, real, constant, α then the packing density is not changed. Intuitively, multiplication of all vectors in the lattice by $\alpha > 0$ means the number of points in a given volume of $\mathbb{R}^n$ will change by a factor $\frac{1}{\alpha^n}$, but the minimum distance of the lattice changes by a factor of α, implying the volume of each ball in the associated packing changes by a factor of α^n. These two effects 'cancel out' so that constant multiplication results in no change in the lattice packing density.

The considerations above motivate the definition of homothecy. Distinct but homothetic lattices are illustrated in Figure 3

Definition 3.5 (Homothecy of Lattice)**.** Lattices Γ_A and Γ_B, with generator matrices A and B respectively are homothetic if there exist $M \in GL_n(\mathbb{Z})$,

$U \in SO_n(\mathbb{R})$ and $\alpha > 0$ such that:

$$B = \alpha UAM$$

For a matrix, X, of determinant $d \neq 0$, multiplication by positive constant $|d|^{-(1/n)}$ gives a matrix, X_1 of determinant ± 1. For the $X_1 = -1$ case, right multiplication by $-I \in GL_n(\mathbb{Z})$, where I is the nxn identity matrix, gives a matrix, X_2 of determinant $+1$. Hence, every invertible matrix is homothetic to at least one matrix of determinant $+1$, i.e. a member of $SL_n(\mathbb{R})$. Therefore, all homothecy classes (i.e. sets of all nxn matrices homothetic to some representative matrix of set) contain members of $SL_n(\mathbb{R})$. This allows homothecy classes to be identified with members of the double coset space:

$$SO_n(\mathbb{R}) \setminus SL_n(\mathbb{R})/GL_n(\mathbb{Z})$$

In the coset space, matrices are regarded as equivalent (i.e. in the same coset) if one can be transformed to the other by left multiplication by members of $SO_n(\mathbb{R})$ and right multiplication by $GL_n(\mathbb{Z})$. Each coset in $SO_n(\mathbb{R}) \setminus SL_n(\mathbb{R})/GL_n(\mathbb{Z})$ is then referred to as a homothecy class of lattice - i.e. it is a set of generator matrices for lattices all of which are homothetic, and thus have the same density of associated sphere packing.

4 The Optimal Lattice Packing in $\mathbb{R}^2$

The definition of homothecy classes for n-dimensional lattices above of course applies to lattices in $\mathbb{R}^2$ so that the homothecy classes of $\mathbb{R}^2$ lattices can be identified with:

$$SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})/GL_2(\mathbb{Z})$$

In what follows, the double coset space is identified with a region of the complex upper half plane, $\mathcal{H}$, in such a way that the complex number $x + yi$ with which a coset is identified encodes information about the packing density of the lattice. It is this encoding which allows the lattice packing of maximum density to be identified. As previously mentioned, this can also be shown the be the densest of all possible packings in $\mathbb{R}^2$, including those not based on lattices, but the proof of that fact is beyond the scope of this report.

4.1 $SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$ and the Upper Half Plane

Theorem 4.1. *Every coset in $SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$ has a unique repersentative in the form:* $y^{-1/2}\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$

Proof. Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Note that $Det(A) = ad - bc = 1$ by definition. For this arbitrary member of the special linear group for 2x2 matrices, it must be shown that A can be transformed into a unique matrix in the form $y^{-1/2}\begin{pmatrix}1 & x\\ 0 & y\end{pmatrix}$ by left multiplication by members of $SO_2(\mathbb{R})$. Note that $SO_2(\mathbb{R})$ may be expressed as:

$$SO_2(\mathbb{R}) = \left\{ S_\theta = \begin{pmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{pmatrix} \ for\, \theta \in [0, 2\pi) \right\}$$

Two cases must be considered.

CASE 1: c = 0

As $Det(A) = 1$, therefore $a = \frac{1}{d}$, and the matrix A has the form:

$$A = \begin{pmatrix} \frac{1}{d} & b \\ 0 & d \end{pmatrix}$$

$S_0 = I, S_\pi = -I \in SO_2(\mathbb{R})$ and so $A_{pos} = \begin{pmatrix} \frac{1}{|d|} & b \\ 0 & |d| \end{pmatrix}$ is a member of the same coset as A. Therefore, letting $y = d^2$ and $x = db$ then gives a member of the coset in the desired form.

CASE 2: $c \neq 0$

Consider a matrix M_θ in the $SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$ coset of A:

$$M_\theta = S_\theta A = \begin{pmatrix} aCos(\theta) - cSin(\theta) & bCos(\theta) - dSin(\theta) \\ aSin(\theta) + cCos(\theta) & bSin(\theta) + dCos(\theta) \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \ 0 \leq \theta < 2\pi$$

Now, as $c \neq 0$:

$$\exists \theta_1 \in (0, \pi) : Cot(\theta_1) = Cot(\pi + \theta_1) = \frac{-a}{c}$$

For $\theta_i = \theta_1, \pi + \theta_1$, clearly $m_3 = 0$. Further, as $\frac{-a}{c} = Cot(\theta_i) = \frac{Cos(\theta_i)}{Sin(\theta_i)}$, $m_1 = \left(\frac{-a^2-c^2}{c}\right) Sin(\theta_i)$. Finally, $SO_2(\mathbb{R}) \subset SL_2(\mathbb{R})$, so $M_{\theta_i} = S_{\theta_i} A \in SL_2(\mathbb{R})$, and hence $Det(M_{\theta_i}) = 1 = m_1 m_4 - m_2 m_3 = m_1 m_4 - 0$, so $m_4 = \frac{1}{m_1}$. Thus for $\theta_i = \theta_1, \pi + \theta_1$:

$$M_{\theta_i} = \begin{pmatrix} m_1 & m_2 \\ 0 & \frac{1}{m_1} \end{pmatrix} = \begin{pmatrix} \left(\frac{-a^2-c^2}{c}\right) Sin(\theta_i) & m_2 \\ 0 & \left(\frac{-c}{a^2+c^2}\right)\left(\frac{1}{Sin(\theta_i)}\right) \end{pmatrix}$$

However, $Sin(\pi + \theta_1) = -Sin(\theta_1)$, which ensures $m_1 > 0$ for exactly one of the two cases $\theta_i = \theta_1, \pi + \theta_1$. Thus, letting $y = m_1^{-2}$ and $x = m_1 m_2$ gives a matrix in the required form.

Hence, in both the $c = 0$ and $c \neq 0$ case, there exists a matrix in the coset of A in the form $y^{-1/2}\begin{pmatrix}1 & x\\ 0 & y\end{pmatrix}$.

Furthermore, this is the unique matrix in the coset with that form, as left multiplication of $M_{\theta i}$ by S_θ for any $\theta \neq 0, \pi$ gives a $(2,1)^{th}$ entry not equal to 0, while multiplication by S_π gives a negative $(1,1)^{th}$ entry. Thus the only possibility for a product in the required form multiplication by S_0, which is the identity matrix $\square$

A consequence of Theorem 4.1 is that the coset space $SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$ is isomorphic with the upper half plane. For $M \in SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$, M contains a unique representative in the form from the statement of the theorem, and identifying the coset with $x + yi \in \mathcal{H}$ defines an isomorphism. Because the homothecy classes for lattices in $\mathbb{R}^2$ have been identified with the double coset space $SO_2(\mathbb{R})\setminus SL_2(\mathbb{R})/GL_2(\mathbb{Z})$, every homothecy class is associated with multiple points in $\mathcal{H}$ via this isomorphism. These are referred to as complex representations of the homothecy class, and given a complex representation $x+yi$, $y^{-1/2}\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$ is a generator matrix in the relevant homothecy class.

4.2 A Fundamental Domain of $\mathcal{H}$

$SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{Z})$ which comprises integer valued 2x2 matrices with determinant +1. Elements of $SL_2(\mathbb{Z})$ can be regarded as representing a function acting on $\mathcal{H}$ as follows:

$$\forall \tau \in \mathcal{H}, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : M(\tau) = \frac{a\tau + b}{c\tau + d}$$

A number of important facts about $SL_2(\mathbb{Z})$ can be proved by basic complex analysis. The proofs are omitted here.

Theorem 4.2.

$$\forall \tau \in \mathcal{H}, M \in SL_2(\mathbb{Z}) : M(\tau) \in \mathcal{H} \tag{1}$$

The proof of Theorem 4.2 relies on showing that if $\tau = x + yi$ with $y > 0$, then for $M \in SL_2((Z))$, with $M(\tau) = w + zi$, $z > 0$. This theorem ensures that $M \in SL_2(\mathbb{Z})$ is a map from $\mathcal{H}$ to $\mathcal{H}$

Theorem 4.3.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \; and \; T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; generate \; SL_2(\mathbb{Z}) \tag{2}$$

Theorem 4.3 means that every $M \in SL_2(\mathbb{Z})$ can be expressed as a product of powers of S and T.

The final key theorem for $SL_2(\mathbb{Z})$ is preceded by a definition.

Definition 4.1 (Fundamental Domain of $\mathcal{H}$)**.** $\mathcal{F}$, the fundamental domain of $\mathcal{H}$ is defined as follows:

$$\mathcal{F} = \{z = x + yi \in \mathcal{H} : |z| = \sqrt{x^2 + y^2} > 1 \; and \; |x| \leq \frac{1}{2}\}$$

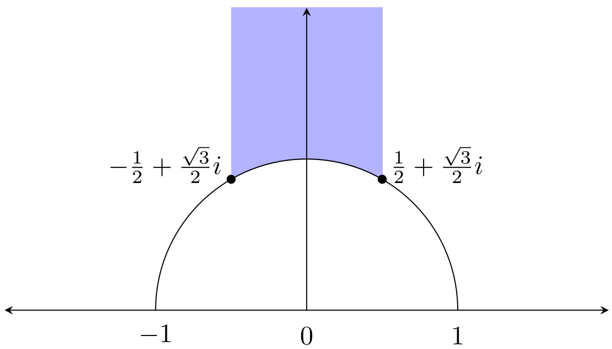

Figure 4: $\mathcal{F}$, the fundamental domain for $\mathcal{H}$

$\mathcal{F}$ is represented graphically in Figure 4.

Theorem 4.4.

$$\forall \tau \in \mathcal{H} : \exists M \in SL_2(\mathbb{Z}) \textit{ such that } M(\tau) \in \mathcal{F} \tag{3}$$

With the above facts, it is possible to prove a theorem related to the complex representations of homothecy classes which is vital to finding the densest lattice packing.

Theorem 4.5. *Every homothecy class has a complex number representative in* $\mathcal{F}$.

Proof. Every member of $\mathcal{H}$ is a complex representation of some homothecy class, and every homothecy class has multiple complex representations in $\mathcal{H}$. By Theorem 4.4, every complex representation can be mapped to $\mathcal{F}$ by a member of $SL_2(\mathbb{Z})$. Therefore, if it can be shown that complex representations related by $SL_2(\mathbb{Z})$ maps on H are homothetic, it then follows that every homothecy class contains a representative whose complex representative is a member of $\mathcal{F}$.

By Theorem 4.3, $SL_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus if the maps associated with these two matrices do not change the homothecy class of a complex representative, then homothecy class does not change when maps of $SL_2(\mathbb{Z})$ are applied.

Let $\tau = x+yi$ be a complex representative of a homothecy class Λ such that:

$$M = y^{1/2} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \in \Lambda$$

Now $S(\tau) = \frac{\tau+1}{1} = (x+1) + yi$ is a complex representative of the lattice with generator matrix:

$$y^{1/2} \begin{pmatrix} 1 & x+1 \\ 0 & y \end{pmatrix} = M.\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

But then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$, and hence the lattice associated with $S(\tau)$ is homothetic to that associated with τ. Hence applying the map on $\mathcal{H}$ associated with S does not change the homothecy class of the complex representation.

Next consider $T(\tau) = \frac{-1}{\tau} = \frac{-1}{x+yi} = \frac{-x+yi}{x^2+y^2}$. This is a complex representation of the homothecy class that contains matrix:

$$N = \left(\frac{y}{x^2+y^2}\right)^{-1/2} \begin{Bmatrix} 1 & \frac{x}{x^2+y^2} \\ 0 & \frac{y}{x^2+y^2} \end{Bmatrix}$$

Now let θ be the angle between vectors $\langle 1, 0\rangle$ and $\langle x, y\rangle$ so that $Cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}$ and $Sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$. Then:

$$S_\theta = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \in SO_2(\mathbb{R})$$

And:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Z})$$

Thus $S_\theta MT$ is homothetic to M and some simple matrix algebra shows $S_\theta MT = N$. Hence, applying the map on $\mathcal{H}$ associated with T also does not change the homothecy class of the lattice represented. It therefore follows that applying $SL_2(\mathbb{Z})$ maps to $\mathcal{H}$ does not change the homothecy class of complex representations, and thus as $\mathcal{F}$ is a fundamental domain of $\mathcal{H}$ for $SL_2(\mathbb{Z})$, thus every homothecy class has a representative in $\mathcal{F}$. □

The importance of Theorem 4.5 lies in the fact that given a lattice with a complex representation in $\mathcal{F}$, the minimum distance for the can easily be identified. Minimum distance in turn plays a key role in determining the density of the associated lattice packing.

4.3 Minimum Distance and Complex Representation

Remark 4.1. Minimum distance, $m(\Gamma)$ of a lattice, Γ, is defined (see Definition 3.4) as

$$m(\Gamma) := \{Min(|\gamma_1 - \gamma_2|)\}_{\gamma_1 \neq \gamma_2 \in \Gamma}$$

However, if Γ comprises all integer linear combination of the basis vectors $\overrightarrow{v_1}, ..., \overrightarrow{v_n}$. Thus if $\gamma_1, \gamma_2 \in \Gamma : m(\Gamma) = \gamma_1 - \gamma_2$, then there exist $x_1, ..., x_n \in \mathbb{Z}$ and $y_1, ..., y_n \in \mathbb{Z}$ such that $\gamma_1 = x_1\overrightarrow{v_1} + ... + x_n\overrightarrow{v_n}$ and $\gamma_2 = y_1\overrightarrow{v_1} + ... + y_n\overrightarrow{v_n}$. Hence:

$$m(\Gamma) = |(x_1 - y_1)\overrightarrow{v_1} + ...(x_n - y_n)\overrightarrow{v_n}| = m(\gamma_3)\ with\ \gamma_3 \in \Gamma$$

Thus the minimum distance for a lattice as defined in Section 3 is equivalent to the minimum Euclidean distance for a vector in Γ.

Theorem 4.6. *For a lattice,* Γ*, with complex representation* $x + yi \in \mathcal{F}$*, the minimum distance of* Γ*,* $m(\Gamma)$ *is* $y^{-1/2}$

Proof. Note that for complex representation $x + yi$, the associated lattice has generator matrix $y^{-1/2}\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{y}} & \frac{x}{\sqrt{y}} \\ 0 & y \end{pmatrix}$

Now from the definition of a generator matrix, the lattice comprises all integer linear combinations of the vectors $\frac{1}{\sqrt{y}}\langle 1, 0\rangle$ and $\frac{1}{\sqrt{y}}\langle x, y\rangle$. Thus:

$$\forall \gamma \in \Gamma : \gamma = \frac{m}{\sqrt{y}}\langle 1, 0\rangle + \frac{n}{\sqrt{y}}\langle x, y\rangle$$

where $m, n \in \mathbb{Z}$.

Hence:

$$|\gamma|^2 = \left(\frac{1}{y}\right)\left((m + nx)^2 + (ny)^2\right) = \left(\frac{1}{y}\right)\left(m^2 + n^2(x^2 + y^2) + 2mnx\right)$$

However, if it is assumed $x + yi \in \mathcal{F}$, then $x^2 + y^2 \geq 1$ and so:

$$|\gamma|^2 \geq \left(\frac{1}{y}\right)\left(m^2 + n^2 + 2mnx\right)$$

Further $x \in [-\frac{1}{2}, \frac{1}{2}]$, so $x > -1$, and thus for $mn \geq 0$:

$$m^2 + n^2 + 2mnx \geq m^2 + n^2 + 2mn(-1) = (m - n)^2$$

On the other hand, $x < 1$, so for $mn \leq 0$:

$$m^2 + n^2 + 2mnx \geq m^2 + n^2 + 2mn(1) = (m + n)^2$$

Therefore in all cases there is an integer, k such that:

$$|\gamma|^2 \geq \left(\frac{1}{y}\right)k^2 \geq \frac{1}{y}$$

And so for all $\gamma \in \Gamma$, $|\gamma| \geq \frac{1}{\sqrt{y}}$. As $|\gamma| = \frac{1}{\sqrt{y}}$ for $m = 1, n = 0$, it follows by Remark 4.1 that this is the minimum distance for Γ, $m(\Gamma)$. $\square$

The minimum distance for a lattice with complex representation in $\mathcal{F}$ is therefore maximized when $y^{-1/2}$ is maximized, thus when y is minimized. The minimum value of y in F is obtained at the points $\pm\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (see Figure 4), giving a maximal minimum distance of:

$$m(\Gamma) = \sqrt{\frac{2}{\sqrt{3}}}$$

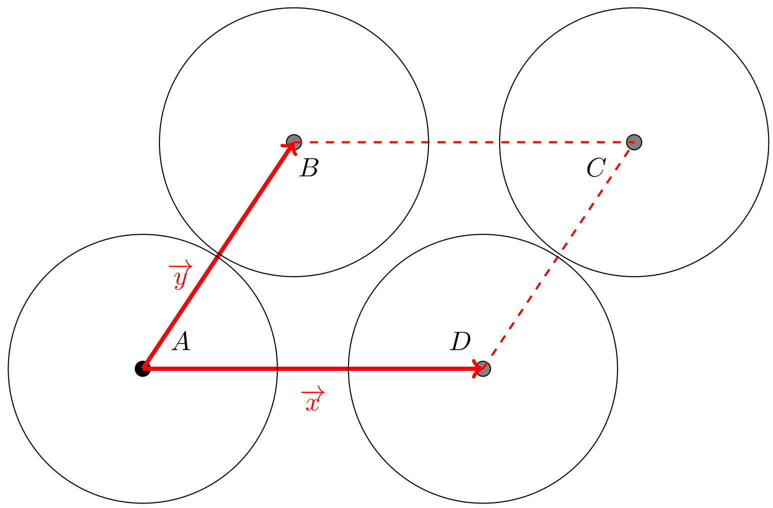

Figure 5: A lattice packing with basis vectors $\overrightarrow{x}$ and $\overrightarrow{y}$. Angles A, B, C, D sum to 2π.

4.4 The Hexagonal lattice packing

Figure 5 illustrates a lattice packing based on a lattice Γ with basis vectors $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$. M, the 2x2 matrix with columns $\overrightarrow{x}$ and $\overrightarrow{y}$ is a generator matrix for the lattice. $\mathbb{R}^2$ can be tiled with parellelograms with sides parallel to $\overrightarrow{x}$ and $\overrightarrow{y}$, each of which intersect with 4 of the circles in the 2 dimensional sphere packing based on the lattice, as in Figure 5. The area of these parallelograms is $det(M)$, and as the angles of a parallelogram sum to 2π, therefore an area equivalent to one of the component circles is covered by the packing in each parallelogram. Hence the lattice packing density is given by the expression:

$$density = \frac{\pi(\frac{m(\Gamma)}{2})^2}{det(M)}$$

It has been shown that every lattice is homothetic to a lattice with complex representation $x + yi \in \mathcal{F}$, and the maximum minimum distance for a lattice with complex representation in $\mathcal{F}$ is:

$$m(\Gamma) = \sqrt{\frac{2}{\sqrt{3}}}$$

Furthermore a lattice with complex representation $x+yi$ has a generator matrix:

$$M = y^{-1/2}\begin{pmatrix}1 & x\\ 0 & y\end{pmatrix} = \begin{pmatrix}\frac{1}{y^{1/2}} & x\\ 0 & y^{1/2}\end{pmatrix}$$

So det(M) = 1. Thus the maximum density for a lattice packing in $\mathbb{R}^2$ is:

$$\frac{\pi}{2\sqrt{3}} \approx 0.9069$$

This density is achieved by lattice with complex representation $x + yi = \pm\frac{1}{2} + \frac{\sqrt{3}}{2}i$, which corresponds to the lattice with basis vectors:

$$\left(\frac{\sqrt{3}}{2}\right)^{-1/2}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\left(\frac{\sqrt{3}}{2}\right)^{-1/2}\begin{bmatrix}\pm\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}$$

or any constant multiple thereof by homothecy. This is the hexagonal lattice packing, in which every circle in the packing is circumscribed by a hexagon (see Figure 1.)

5 Theta Functions

5.1 Motivation and Definition

In the previous section the densest lattice packing in $\mathbb{R}^2$ was identified. However, the method by which this was achieved relied on several characteristics specific to lattices in $\mathbb{R}^2$ and cannot be generalized easily to higher dimensions. In particular, the method of finding the minimum distance was based on a relationship between $SO_2(\mathbb{R}) \setminus SL_2(\mathbb{R})$ and the complex upper half plane which clearly does not apply to matrices of dimension greater than 2. Finding the minimum distance for a lattice in $\mathbb{R}^n$ for $n > 2$ is therefore key to solving the sphere packing problem in higher dimensions. Theta functions are a method of studying the distribution of lengths of vectors in a lattice and so can be applied to the study of minimum distance of a lattice (by Remark 4.1) [Elk00]

Definition 5.1 (Theta function of a lattice)**.** $\Theta_\Gamma(z)$, the theta function of a lattice, Γ is the generator function of the squared lengths of vectors in that lattice. Thus:

$$\Theta_\Gamma(z) = \sum_{x\in\Gamma} z^{(x,x)}$$

For $\tau \in \mathcal{H}$, there are some nice identities for lattice theta function composed with the function $f(\tau) = e^{\pi i \tau}$. It is therefore useful to define a modified version of the theta function, as follows:

Definition 5.2 (Modified Theta Function)**.** For $\tau \in \mathcal{H}$ and lattice $\Gamma \subset \mathbb{R}^2$, define:

$$\theta_\Gamma(\tau) = \Theta_\Gamma(e^{\pi i \tau}) = \sum_{x\in\Gamma} e^{\pi i (x,x)\tau}$$

5.2 Lattice Duals

Definition 5.3 (Dual of a Lattice)**.** Given $C \subset \mathbb{R}^n$, a lattice, the dual of C, $C*$, is the set of all vectors $y \in \mathbb{R}$ such that $(y, x) \in \mathbb{Z}$ for all $x \in C$

It can be shown that for any lattice Γ, $\Gamma*$ is also a lattice and a generator matrix for $C*$ can easily be described given a generator matrix of Γ,

Theorem 5.1. *Let $\Gamma \in \mathbb{R}^n$ be a lattice with generator matrix A. Then $C*$ is also a lattice in $\mathbb{R}^n$ with generator matrix $(A^T)^{-1}$ (the inverse of the transpose of A).*

Proof. Consider $\Gamma \in \mathbb{R}^n$ be a lattice with generator matrix A, and $y \in \Gamma* \subset \mathbb{R}^n$. Let the columns of A be $\overrightarrow{a_1}, ..., \overrightarrow{a_n}$. It is clear that y is in $\Gamma*$ if an only if $y.\overrightarrow{a_i} \in \mathbb{Z}$ for $i = 1, ..., n$. Thus:

$$\begin{pmatrix} y.\overrightarrow{a_1} \\ \vdots \\ y.\overrightarrow{a_n} \end{pmatrix} = \overrightarrow{z} \in \mathbb{Z}^n$$

But is is clear from matrix algebra that this means:

$$A^T y = \overrightarrow{z}$$

$$\Rightarrow y = (A^T)^{-1}\overrightarrow{z}$$

Hence $\Gamma*$ comprises precisely those vectors in $\mathbb{R}^n$ which are integer linear combinations of the columns of $(A^T)^{-1}$. But by definition this means $C*$ is a lattice with $(A^T)^{-1}$ a generator matrix. □

The fact the dual of a lattice is itself a lattice is important for another definition which is significant in discussion of lattices and their theta functions.

Definition 5.4 (Self Dual Lattice)**.** A lattice $\Gamma \in \mathbb{R}^n$ is self dual if $\Gamma* = \Gamma$.

There is a simple consequence of this definition and Theorem 5.1 which is of relevance later.

Theorem 5.2. *For a self dual lattice with generator matrix A, $Det(A) = \pm 1$*

Proof. By Theorem 5.1, for latice Γ with generator matrix A, the generator matrix of $\Gamma*$, the dual of Γ is $(A^T)^{-1}$. Thus if Γ is self dual, then by Theorem 3.1 there exists $M \in GL_n(\mathbb{Z})$ such that:

$$(A^T)^{-1} = AM$$

But $det(M) = 1$, so:

$$det((A^T)^{-1}) = det(A)$$

Hence:

$$\frac{1}{Det(A^T)} = \frac{1}{Det(A)} = Det(A)$$

But this implies $Det(A) = \pm 1$, as required. □

It is not immediately obvious why self-dual lattices should be of particular interest, given they appear to have a rather complicated defining property which should not apply to more than a small fraction of all possible lattices. There are two reasons why study of self dual lattices is useful. Firstly, many of the most interesting and widely studied are in fact self dual. For example, the lattice in $\mathbb{R}^n$ consisting of all points with integer coordinates - in some sense the prototypical example - is clearly self dual. The E8 lattice in eight dimensions is a more complicated example of self duality which will be discussed further below. Secondly the modified theta functions of self dual lattices have some particularly nice properties which facilitate their study. Two particularly important identities for self-dual lattices are the following:

Theorem 5.3 (Self Dual Lattice Identity 1)**.** *For self dual lattice* Γ *and* $\tau \in \mathcal{H}$*:*

$$\theta_\Gamma(\tau+2) = \theta_\Gamma(\tau)$$

Theorem 5.4. *[Self Dual Lattice Identity 2] For self dual lattice* Γ *and* $\tau \in \mathcal{H}$*, there exists an eighth root of unit,* ϵ_g*, such that:*

$$\theta_\Gamma\left(-\frac{1}{\tau}\right) = (\epsilon_g \tau^{1/2})^n \theta_\Gamma(\tau)$$

The proof of Theorem 5.3 is simple. If Γ is self dual, then for all $x \in \Gamma$, (x,x) is an integer. Then by definition 5.2:

$$\theta_\Gamma(\tau+2) = \sum_{x\in\Gamma} e^{(\pi i(x,x))(\tau+2)} = \sum_{x\in\Gamma} e^{2\pi i(x,x)} e^{\pi i(x,x)\tau} = \sum_{x\in\Gamma} e^{\pi i(x,x)\tau} = \theta_\Gamma(\tau)$$

Theorem 5.4 is considerably more difficult to prove, and requires the application of a result from Fourier analysis known as the Poisson Summation Formula for Lattices.

5.3 The Poisson Summation Formula for Lattices

The Poisson Summation Formula for Lattices relates to the theory of Fourier transforms.

Definition 5.5 (Fourier Transform)**.** For an integrable function $f\mathbb{R}^n \to \mathbb{C}$, the Fourier transform of f, $\hat{f}$ is defined:

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{2\pi i(x,t)} dt$$

In particular, for $n = 1$:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{2\pi i x t} dt$$

The formulas discussed in this section to apply to a function f, it must be a Schwartz function

Definition 5.6 (Schwarz Function)**.** A function $f\mathbb{R}^n \to \mathbb{C}$ is a Schwarz function if it is infinitely differentiable (smooth) and if it exhibits rapid decay such that:

$$|f(x)| << |x|^{-N} \, as \, x \to \infty \, for \, all \, N$$

The Poisson Summation Formula for Lattices relies on a simpler and more familiar result - the Poisson Summation Formula for the Real Numbers.

Theorem 5.5 (Poisson Summation Formula for $\mathbb{R}$)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwarz function. Then:*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Proof. The proof of this theorem is based on lecture notes by Henri Darmon, available online at `https://www.math.mcgill.ca/darmon/courses/11-12/nt/notes/lecture3.pdf`. [DC11]

Consider Schwarz function $f : \mathbb{R} \to \mathbb{C}$ and let $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$. Clearly F is periodic and can be expressed as a Fourier series expansion:

$$F(x) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m x}$$

Where:

$$a_m = \int_0^1 F(x) e^{-2\pi i m x} dx$$

Thus by definition:

$$a_m = \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n) e^{-2\pi i m x} dx = \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m x} dx = \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{2\pi i m x} dx$$

If $t = x + n$, then for all $n \in \mathbb{Z}$ $e^{2\pi i m x} = e^{2\pi i m(-n)} e^{2\pi i m t} = (1) e^{2\pi i m t}$, and so:

$$a_m = \sum_{n \in \mathbb{Z}} \int_{-n}^{-n+1} f(t) e^{2\pi i m t} dt = \int_{-\infty}^{\infty} f(t) e^{2\pi i m t} dt$$

So, by Definition 5.5:

$$F(x) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m x} = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i m x}$$

By definition, $F(x) = \sum_{m \in \mathbb{Z}} f(x+m)$, and so:

$$\sum_{m \in \mathbb{Z}} f(x+m) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i m x}$$

The theorem follows by letting $x = 0$.

□

Theorem 5.6 (Poisson Summation Formula for Lattices)**.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Schwarz function and $\Gamma \in \mathbb{R}^n$ be a lattice with generator matrix A and dual lattice $\Gamma*$. Then:*

$$\sum_{x\in\Gamma} f(x) = \frac{1}{det(A)} \sum_{x\in\Gamma*} \hat{f}(x)$$

Proof. The proof of this theorem is based on lecture notes by Michael Magee, available online at `https://gauss.math.yale.edu/~mrm89/lecture7.pdf`. [Mag16]

Let $\Gamma \subset \mathbb{R}^n$ be a lattice with generator matrix A such that the columns of A are $\overrightarrow{a_1}, ..., \overrightarrow{a_n}$. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Schwarz function. From the definition of a lattice:

$$\sum_{x\in\Gamma} f(x) = \sum_{m_1\in\mathbb{Z}} ... \sum_{m_1\in\mathbb{Z}} f(m_1\overrightarrow{a_1}, ..., m_n\overrightarrow{a_n})$$

For $\overrightarrow{z} = (m_1, ..., m_n) \in \mathbb{Z}^n$, $m_1\overrightarrow{a_1} + ... + m_n\overrightarrow{a_n} = A\overrightarrow{z}$, so that

$$\sum_{x\in\Gamma} f(x) = \sum_{\overrightarrow{z}\in\mathbb{Z}} f \circ A(\overrightarrow{z})$$

Theorem 5.5 can be applied n times which shows:

$$\sum_{x\in\Gamma} f(x) = \sum_{\overrightarrow{z}\in\mathbb{Z}} \widehat{f \circ A}(\overrightarrow{z})$$

From the definition of Fourier Transform:

$$\widehat{f \circ A}(\overrightarrow{z}) = \int_{\mathbb{R}^n} f(A(t))e^{2\pi i(z,t)}dt$$

Letting $u = A(t)$, the above integral is transformed to:

$$\widehat{f \circ A}(\overrightarrow{z}) = (det(A))^{-1} \int_{\mathbb{R}^n} f(u)e^{2\pi(z,A^{-1}u)}du$$

Basic linear algebra can be used to show $(\overrightarrow{v}, M\overrightarrow{u}) = (M^T\overrightarrow{v}, \overrightarrow{u}$ for $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$ and nxn matrix M. Thus:

$$\widehat{f \circ A}(\overrightarrow{z}) = (det(A))^{-1} \int_{\mathbb{R}^n} f(u)e^{2\pi((A^{-1})^T z,u)}du$$

Again by basic linear algegra, $(A^{-1})^T = (A^T)^{-1}$, so by Theorem 5.1, $(A^{-1})^T$ is the generator matrix for $\Gamma*$, the dual of Γ, from which it follows that:

$$\sum_{\overrightarrow{z}\in\mathbb{Z}} \widehat{f \circ A}(\overrightarrow{z}) = \sum_{x\in\Gamma*} \hat{f}(x)$$

And therefore:

$$\sum_{x\in\Gamma} f(x) = \sum_{x\in\Gamma*} \hat{f}(x)$$

□

The analytic conditions for f and f in Theorem 5.5 are automatically satisfied by lattice theta functions. As a rigorous proof of this would require an extended treatment of Fourier analysis this fact is assumed true here and Theorem 5.5 is applied to theta functions.

5.4 Proof of Self Dual Lattice Identity 2

The proof of Theorem 5.4 requires two lemmas related to Fourier Transforms.

Lemma 5.1. *For Schwarz function $f : \mathbb{R}^n \to \mathbb{C}$, if $k \neq 0 \in \mathbb{R}$, then:*

$$\hat{f}(kx) = \frac{1}{k^n} \hat{f}\left(\frac{x}{k}\right)$$

Proof. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Schwarz function and $g(x) = f(kx)$ and $k \neq 0$:

$$\hat{f}(kx) = \int_{\mathbb{R}^n} f(kt) e^{2\pi i (x,t)} dt$$

Letting $u = kt$ and recalling that u and t are n-dimensional vectors, the integral variable can be changed as follows:

$$\hat{f}(kx) = \int_{\mathbb{R}^n} \left(\frac{1}{k^n}\right) f(u) e^{2\pi i (x, \frac{u}{k})} du = \int_{\mathbb{R}^n} \left(\frac{1}{k^n}\right) f(u) e^{2\pi i (\frac{x}{k}, u)} du$$

The lemma follows by Definition 5.5 $\square$

Lemma 5.2. *For function $f : \mathbb{R}^n \to \mathbb{C}$ such that $f(x) = e^{-\pi(x,x)}$, $\hat{f}(x) = f(x)$*

Proof. Let $f(x) = f(x) = e^{-\pi(x,x)}$. Then:

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i (x,t)} e^{\pi(t,t)} dt = \int_{\mathbb{R}^n} e^{-\pi((t-ix)^2 + x^2)} = e^{-\pi x^2} \int_{\mathbb{R}^n} e^{-\pi((t-ix)^2}$$

Note that if $\overrightarrow{x} = (x_1, ..., x_n)$ and $\overrightarrow{t} = (t_1, ..., t_n)$, then:

$$\int_{\mathbb{R}^n} e^{-\pi((t-ix)^2} = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-\pi(t_1 - ix_1)^2} ... e^{-\pi(t_b - ix_n)^2} dt_1 ... dt_n$$

So the lemma will be proved if it can be shown $\int_{-\infty}^{\infty} e^{-\pi(t_k - ix_k)^2} dt_k = 1$ for $k = 1, ..., n$. Letting $z = t_k - ix_k$. Rewriting the integral on the left hand side of this equation gives:

$$\int_{-\infty}^{\infty} e^{-\pi(t_k - ix_k)^2} dt_k = \int_{\mathbb{R}+ix_k} e^{-\pi z^2} dz$$

Now, $f(x) = e^{-\pi z^2}$ is holomorphic, so by Cauchy's integral theorem, the integral of $f(x)$ along any simple closed curve in $\mathbb{C}$ is 0. In particular, this applies to the curve in red in Figure 6. Further, as M approaches infinity

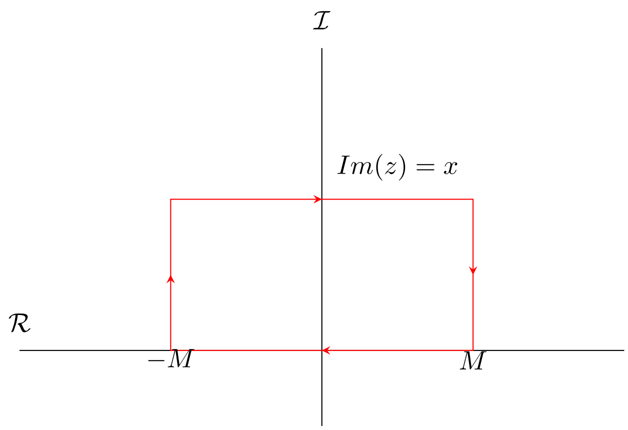

Figure 6: Cauchy's Integral Theorem implies integral along the horizontal sections of the curve are equal.

in that figure, the intrgral of $e^{-\pi z^2}$ on the parts of the curve parallel to the imaginary axis approach 0. From this it follows:

$$\int_{\mathbb{R}+ix_k} e^{-\pi z^2} dz = \int_{\mathbb{R}} e^{-\pi z^2} dz$$

Hence the lemma will be proved if it can be shown $\int_{\mathbb{R}} e^{-\pi z^2} dz = 1$. But this is true by the following calculation:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\pi z^2} dz &= 2\int_0^\infty e^{-\pi z^2} dz \\
&= 2\sqrt{\int_0^\infty e^{-\pi u^2} du \int_0^\infty e^{-\pi w^2} dw} \\
&= 2\sqrt{\int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} re^{-\pi r^2} d\theta dr} \\
&= 2\sqrt{\frac{\pi}{2} \int_{r=0}^\infty re^{-\pi r^2} dr}
\end{aligned}$$

Integrating using the substitution $u = r^2$ shows:

$$\int_{r=0}^\infty re^{-\pi r^2} dr = \frac{1}{2\pi}$$

Thus $\int_{\mathbb{R}} e^{-\pi z^2} dz = 1$ and the lemma is proved. □

With the two above lemmas and the Poisson summation formula for lattices the proof of Theorem 5.4 is quite straightforward. For $\tau \in \mathcal{H}$, let $f_\tau(x) =$

$e^{-\pi(x,x)\tau}$ for $x \in \mathbb{R}^n$. Then if $\Gamma \in \mathbb{R}^n$ is a lattice:

$$\theta_\Gamma(\tau) = \sum_{x\in\Gamma} f_\tau(\sqrt{-i\tau}x)$$

By Theorem 5.6, if A is a generator matrix of Γ, and $\Gamma*$ is the dual of Γ:

$$\theta_\Gamma(\tau) = \frac{1}{det(A)} \sum_{x\in\Gamma*} \hat{f}_\tau(\sqrt{-i\tau}x)$$

If Γ is self dual, then $\Gamma = \Gamma*$ and so by Lemma 5.1:

$$\theta_\Gamma(\tau) = \frac{1}{\left(\sqrt{-\tau i}\right)^n det(A)} \sum_{x\in\Gamma} \hat{f}_\tau \left(\frac{x}{\sqrt{-i\tau}}\right)$$

By Theorem 5.2, $det(A) = 1$. Also, by Lemma 5.2, $\hat{f}(x) = f(x)$. Thus:

$$\left(\sqrt{-i}\sqrt{\tau}\right)^n \theta_\Gamma(\tau) = \sum_{x\in\Gamma} f_\tau \left(\frac{x}{\sqrt{-i\tau}}\right)$$

Considering the right hand side of the equation:

$$\sum_{x\in\Gamma} f_\tau \left(\frac{x}{\sqrt{-i\tau}}\right) = \sum_{x\in\Gamma} e^{-\pi\left(-\frac{1}{i\tau}\right)(x,x)}$$

Because $\frac{1}{i} = -i$, it follows:

$$\sum_{x\in\Gamma} f_\tau \left(\frac{x}{\sqrt{-i\tau}}\right) = \sum_{x\in\Gamma} e^{\pi i\left(\frac{-1}{\tau}\right)(x,x)} = \theta_\Gamma\left(\frac{-1}{\tau}\right)$$

Letting $\epsilon_g = \sqrt{-i}$, an eighth root of unity, it therefore follows:

$$\theta_\Gamma\left(\frac{-1}{\tau}\right) = \left(\epsilon_g\sqrt{\tau}\right)^n \theta_\Gamma(\tau)$$

which is Lemma 5.4

5.5 Modular Forms and Theta Functions

In section 4.3, the matrices in $SL_2(\mathbb{Z})$ were interpreted as maps acting on the complex upper half plane $\mathcal{H}$ such that:

$$For\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : M(\tau) = \frac{a\tau + b}{c\tau + d}$$

In particular, $\lambda_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ correspondes to the map $\tau \to \tau + 2$ and $\lambda_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ correspondes to the map $\tau \to -\frac{1}{\tau}$. Hence, from the identities stated in Section 5.2 for a self-dual lattice $\Gamma \subset \mathbb{R}^n$:

$$\theta_\Gamma(\lambda_1(\tau)) = \theta(\tau) \tag{4}$$

and (for ϵ_g an eighth root of unity):

$$\theta_\Gamma(\lambda_2(\tau)) = (\epsilon_g \tau^{1/2})^n \theta(\tau) \tag{5}$$

The matrices λ_1 and λ_2 generate a subgroup of $SL_2(\mathbb{Z})$, denoted Λ, which comprises all matrices congruent mod 2 to either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda$, equations (4) and (5) imply:

$$\theta_\Gamma(M(\tau)) = (\epsilon_g (c\tau + d)^{1/2})^n \theta_\Gamma(\tau) \tag{6}$$

This result provides the link between lattice theta functions and modular forms.

Definition 5.7 (Modular Form)**.** [Coh18] A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of level Λ (for Λ subgroup of $SL_2(\mathbb{Z})$) and weight k if

1. For all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda : f(M(\tau)) = (c\tau + d)^k f(\tau)$
2. f is holomorphic on $\mathcal{H}$
3. $|f(\tau)|$ remains bounded as $Im(\tau) \to \infty$

Points 2 and 3 of the definition above can be shown to hold for any theta function. Considering Equation (6) for the theta function of a self dual lattice, $\Gamma \in \mathbb{R}^n$, as ϵ_g is an eighth root of unity, it is clear that if n is divisible by 8, then θ_Γ is a modular form of level Λ and weight $\frac{n}{2}$. One such lattice is E8, a lattice in $\mathbb{R}^8$ comprising all points in $\mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8$ such that the sum of their coordinates is even. E8 is described as an even lattice, meaning the squared Euclidean length is an even integer for every member of the lattice. A consequence of this is that θ_{E8}, is a 'classic' modular form such that the functional equation holds for all any $M \in SL_2(\mathbb{Z})$. This follows from the fact that for an even lattice, if $\lambda_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\theta_\Gamma(\lambda_3(\tau)) = \theta_\Gamma(\tau)$, and λ_2, λ_3 are generators for $SL_2(\mathbb{Z})$. E8 has 240 roots or minimal non-zero length vectors and thousands of symmetries among its roots. Figure 7 is a reproduction of Peter McMullen's drawing of 2-dimensional representation of the E8 root system which captures some of the symmetries.

This report began with a description of the sphere packing problem as a motivation for the study of lattices, and it was shown in Section 4 how properties of lattices can be exploited to identify a very dense packing in 2 dimensions which is the optimal possible lattice packing for $n = 2$. The E8 lattice is also intimately connected with sphere packing. The lattice packing associated with $E8$ was long suspected to be the densest in 8 dimensions and this was proved by Maryna Viazovska in 2017 (an achievement which led to the awarding of a Fields Medal in 2022). [Via17] Remarkably, this was the same year of the publication of the definitive proof of the optimal packing in 3 dimensions [Hal+17], a problem

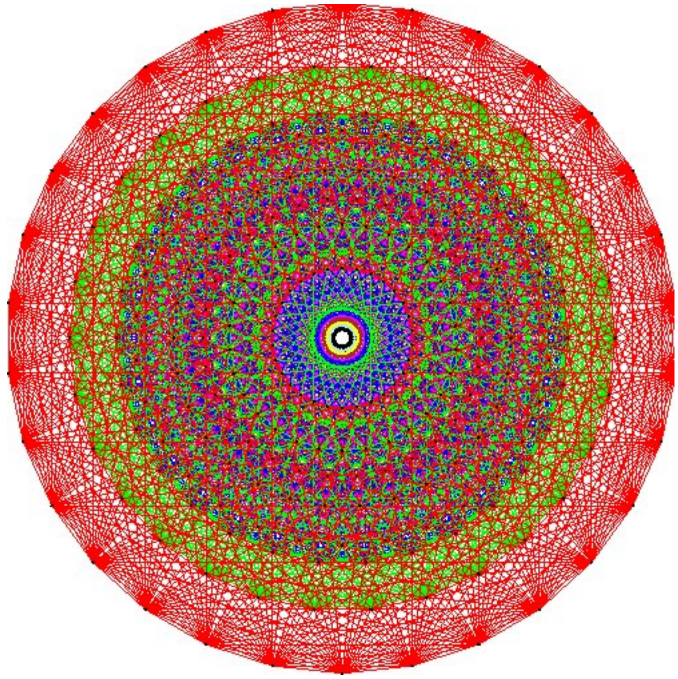

Figure 7: 2D representation of E8, from: `https://aimath.org/E8/mcmullen.html`

which can be described more intuitively and might have been expected to have a much simpler solution. This reflects in part the applicability of the mathematical machinery of modular forms to the theta function of the E8 lattice.

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