Examining periodic solutions to the N-Body problem using computational methods

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What is the aim of my project?

- To write a program that models the motion of N-bodies
- Experiment with different initial conditions to create stable or periodic orbits
- Analyse how small perturbations effect these periodic orbits

Equations of Motion

The equation of motion for the system is derived from Newton's Law of gravitation:

$$\mathbf{a_i} = G \sum_{i \neq j}^{N} \frac{m_j \mathbf{r}}{|\mathbf{r}|^3}$$

Where $\mathbf{a_i} = (a_{ix}, a_{iy}, a_{iz})$, $m_j = mass\ of\ body\ j$, $\mathbf{r} = \mathbf{q_i} - \mathbf{q_j}$ and $\mathbf{q} = (q_x, q_y, q_z)$

The python code

- By converting MATLAB code found online [1] to Python using ChatGPT and by altering it to my needs, I was able to plot the orbits of a system of N-bodies from initial conditions and using numerical integration methods.
- By inputting the initial positions and velocities of each object as a set of arrays, the system can be plotted as an image or GIF.

Constraints and assumptions

The code used makes a few assumptions about the system:

- Each body is a small, smooth uniform sphere.
- The Gravitational constant, G, is taken as $G = 1$ [2]
- The system is only modelled in the 2D plane.[3]

Leapfrog Integration

- The method used for integrating is the Leapfrog Method. It takes the following form for integrating our equation:

$$v_{i+\frac{1}{2}} = x_i + a_i \frac{\Delta t}{2}$$

$$x_{i+1} = x_i + v_{i+\frac{1}{2}} \Delta t$$

$$v_{i+1} = v_{i+\frac{1}{2}} + a_{i+1} \frac{\Delta t}{2}$$

- This method is a better choice than other methods such as Runge-Kutta as it is time-reversible and tends to drift less, however, drift still occurs over larger time-scales [4]

Chaotic System

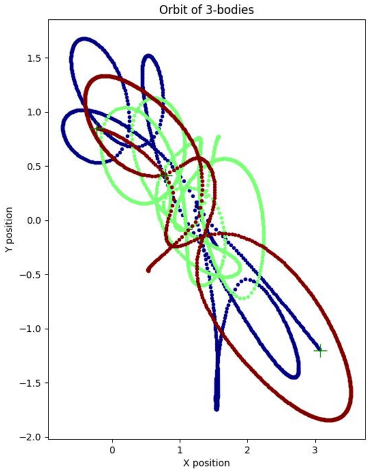


Figure: Orbit of 3 random mass bodies with random initial conditions

Pre-existing solutions

- In 1772, **Joseph-Louis Lagrange** proposed the equillateral triangle solution to the 3-body problem. [5]
- In 1985, **Perko and Walter** showed that for $N \geq 4$ in a central configuration it is periodic.[6]
- **Milovan Šuvakov** created a gallery online of conditions for periodic 3-Body orbits. [7]

Figure of Eight

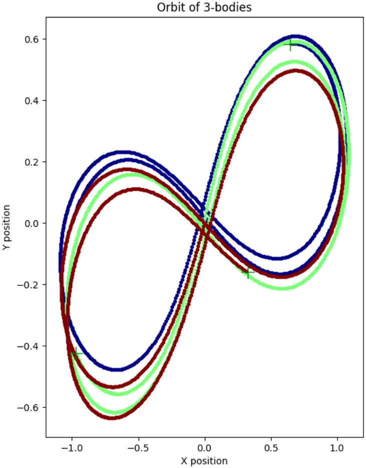


Figure: Periodic "Figure of Eight" Solution

Other Systems that are periodic

- Starting with 3 bodies I began searching for other systems that would produce periodic orbits.
- The natural first step was to position 3 bodies in a triangle shape.
- By placing them in this configuration and setting these initial velocities the following orbit is produced:

Triangle Initial Positions

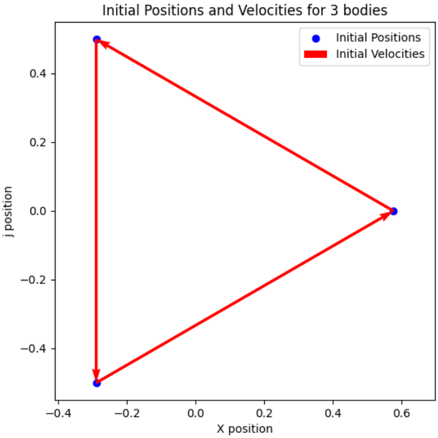


Figure: Triangle Initial Conditions

Triangle Orbit

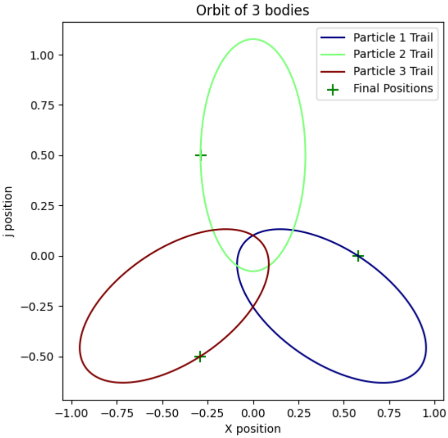


Figure: Triangle Orbit

Theory

From examining various shapes of initial positions I came to the following conclusion:

Theory

Given N number of identical bodies, placing them on the vertices of an N sided polygon, setting the magnitude of each velocity to be the same and setting the direction of initial motion to be pointing at the same angle to each other, a periodic orbit will be formed.

The Proof!

Proof.

Using the equation of Force:

$$\mathbf{F_i} = Gm_i \sum_{i \neq j}^{N} \frac{m_j \mathbf{r}}{|\mathbf{r}|^3}$$

Since,

$$m_i = m_j = \frac{\sum_i^N m_i}{N}$$

$$\therefore \sum_{i \neq j}^{N} m_j = M_j = const.$$

$\square$

Proof continued

Proof.

Similarly, since all bodies positioned on vertices of a regular N-polygon:

$$R(t) = r_j - r_i$$

Since it is an N-polygon:

$$|R(t)| = constant \; \forall \; i, j \; such \; that \; i \neq j$$

$$\implies F_i(t) = GM^2 \frac{R(t)}{|R(t)|^3}$$

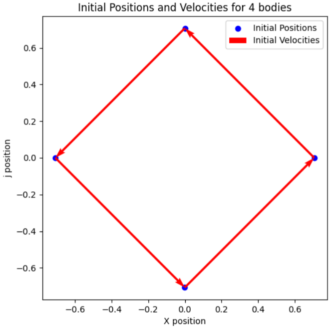
$$\therefore |F_i| = |F_1| = |F_2| = \cdots = |F_N|$$

All bodies experience the same magnitude of force as every other body at all times provided no external influence, the magnitudes of initial velocities are the same and the direction of velocity is at the same relative angle for each body. □

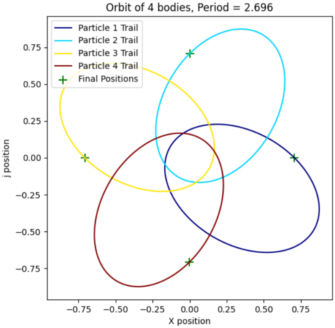
Proving using computational methods

- By adding to the code, it became possible to estimate the period of orbit of the system.
- The exact period cannot be calculated as this is still all done numerically, so drift occurs.
- This could be used to show using computational methods that this theory holds true for all tested values of $N \geq 3$

Examples for $N \geq 4$

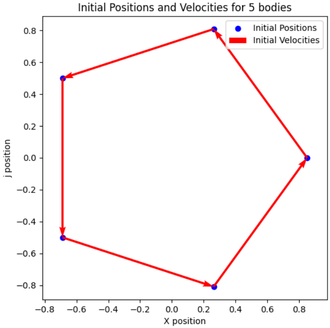


(a) Initial

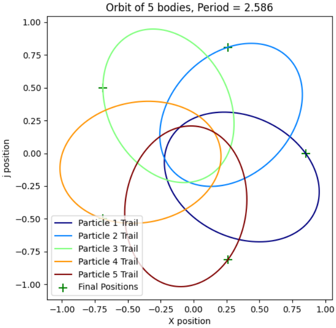


(b) Orbit

continued.

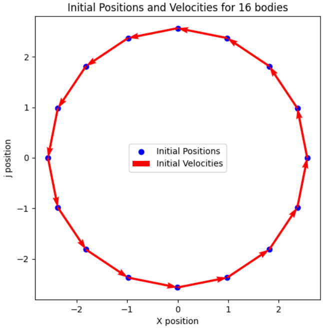


(a) Initial

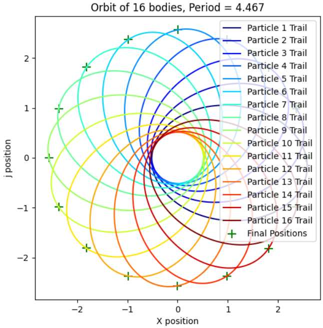


(b) Orbit

continued.



(a) Initial

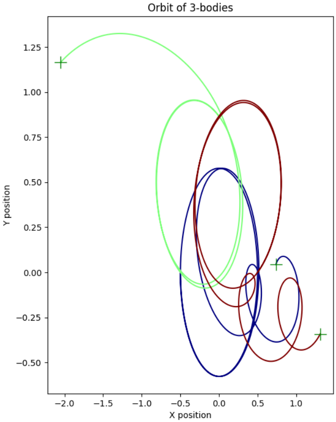


(b) Orbit

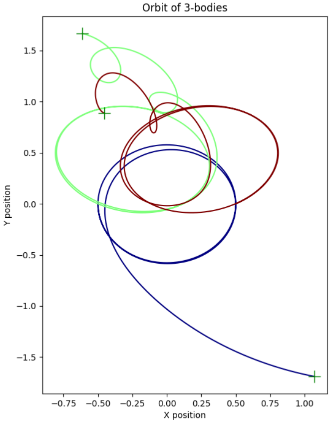
A note on stability in Z

- At the beginning I mentioned the orbital plane and my 2D plotting.
- The following are examples of the instability of the system in X-Y and the stability of it in Z by applying the same perturbation each time but on different axes.

$N = 3$ Stability

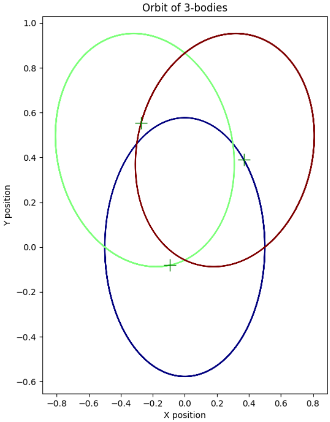


(a) Perturbation in X

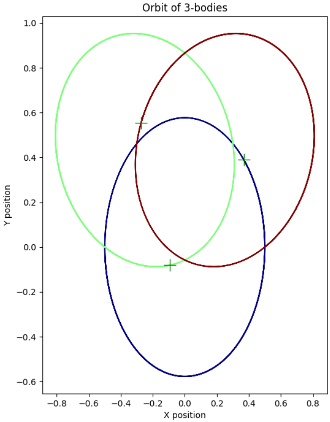


(b) Perturbation in Y

$N = 3$ Stability

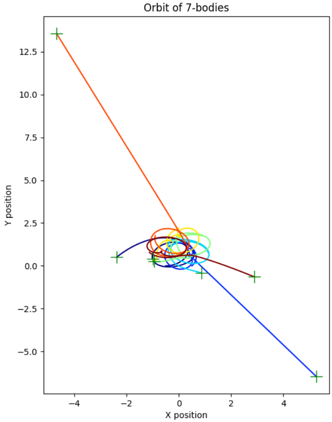


(a) No Perturbation

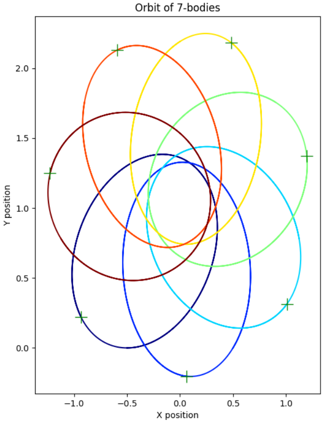


(b) Perturbation in Z

$N > 3$ Stability



(a) Perturbation in X



(b) Perturbation in Z

What's next?

Further research options in this area include:

- Use 3D polygons for initial conditions
- Consider relativistic effects
- Consider non-spherical bodies
- Consider the electrostatic N-body problem using Coulombs law

Conclusion

- A python program can be used to accurately model the N-Body problem.
- Periodic orbits can be formed using the polygon method mentioned.
- The periods can be verified using the python code.
- The system only needs to be modelled in two dimensions.

[1] Yash. *The N-Body Problem*. URL: https://medium.com/quantaphy/the-n-body-problem-2acda67b11b5.

[2] Matthew Choptuik. *PHYS 210: Introduction to Computational Physics Finite Difference Solution of N -Body Problems*. http://bh0.physics.ubc.ca/People/matt/Teaching/12Fall/PHYS210/ 2015.

[3] Kyle T. Alfriend et al. "Chapter 2 - Fundamental Astrodynamics". In: *Spacecraft Formation Flying*. Oxford: Butterworth-Heinemann, 2010, pp. 13–38. ISBN: 978-0-7506-8533-7. URL: https://www.sciencedirect.com/science/article/pii/B9780750685337002074.

[4] L.F. Shampine. "Stability of the leapfrog/midpoint method". In: *Applied Mathematics and Computation* (). URL: https://www.sciencedirect.com/science/article/pii/S0096300308008758.

[5] Wikipedia contributors. *N-body problem — Wikipedia, The Free Encyclopedia*. https://en.wikipedia.org/w/index.php?title=N-body_problem&oldid=1229503888. [Online; accessed 21-June-2024]. 2024.

[6] L. M. Perko and E. L. Walter. "Regular Polygon Solutions of the N-Body Problem". In: *Proceedings of the American Mathematical Society* 94.2 (1985), pp. 301–309. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2045395 (visited on 06/22/2024).

[7] Milovan Šuvakov. *Periodic orbits of the three body problem*. URL: http://three-body.ipb.ac.rs/.